ON ORBITS UNDER ERGODIC MEASURE-PRESERVING TRANSFORMATIONS

BY
DOROTHY MAHARAM (1)

1. **Introduction.** If a transformation T (from a space to itself) is given, two natural questions arise. First, what properties does the orbit(2) of a general point have? Secondly, to what extent does the general orbit determine T(3)? When T is an ergodic measure-preserving transformation of the unit interval, we shall obtain a partial answer to the first question, and an answer to the second (Theorem 5, 7.3). We show, in fact, that the orbit of almost every point determines T almost everywhere, in a "natural" way. Here "natural" requires some comment. If T were continuous (a. e.), it would be determined (a. e.) by the orbit ξ (where $\xi(n) = T^n(x)$) by the simple rule

$$\xi(n_i) \to y \text{ implies } \xi(n_i + 1) \to T(y). \tag{1}$$

Though this is too simple to work in general (even when considerably generalized, as we show by an example in 7.4), we shall give a generalized rule for determining T from ξ; the method depends ultimately on the fact that (from Lusin's theorem) T will be continuous if we remove sets of small measure.

This means that, in a sense, the study of an ergodic measure-preserving transformation T of the unit interval can be reduced to the study of a single sequence of points (the orbit of a "general" point). As an application of this point of view, we obtain sequence-characterizations of T being weakly or strongly mixing (Theorem 6, 8.2).

In more detail, we proceed as follows. After some sections on notation, and on reliminary results concerning uniformly distributed sequences, and measure-preserving and ergodic transformations, we introduce (§5) the general notion of a "D-sequence," and prove (Theorem 1, 5.2) that each D-sequence determines, by an appropriate generalization of (1), an essentially unique transformation T

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(2) Throughout, we use "orbit" as short for "positive semi-orbit"; thus the "orbit" of x under T is the sequence $\{x, Tx, T^2x, \ldots\}$.

(3) Perhaps even more natural questions are: What properties does the orbit of an arbitrary point have, and to what extent does an arbitrary orbit determine T? But these questions have the trivial answer "none," since T can be altered arbitrarily on any one sequence of points, remaining ergodic and measure-preserving.

(not necessarily ergodic or measure-preserving), of which it is an orbit. If the D-sequence is uniformly distributed, T will be measure-preserving (Theorem 2, 6.2), and is not affected if the sequence is altered on a set of integers of density 0 (Theorem 3, 6.3); but it need not be ergodic (6.4). By imposing further requirements on the sequence, we define the notion of "E-sequence" (§7), and prove that each E-sequence is a uniformly distributed D-sequence, and that the transformation it determines is ergodic and measure-preserving (Theorem 4, 7.2). Conversely (Theorem 5, 7.3), if T is any ergodic measure-preserving transformation, then the orbit of almost every point under T is an E-sequence, and the transformation it determines coincides (a.e.) with T. Finally, in Theorem 6 and its corollary (8.2, 8.3) we extend these results to characterize strongly and weakly mixing transformations.

Throughout, we consider only transformations of the *unit interval* I (see 4.1 for the precise definition of "transformation"). The results could easily be extended to somewhat more general situations, but we do not go into this. However, the *compactness* of I is used quite freely, so that the extension of the results to the real line would already present new problems.

2. **Notation and preliminaries.**

2.1. *Sets of integers.* Letters like m, n, k denote integers, usually non-negative. We write $\mathscr{N} = \{0, 1, 2, \cdots\}$, $\mathscr{N}_k = \{0, 1, \cdots, k\}$. Script letters $\mathscr{M}$, $\mathscr{P}$, etc., denote subsets of $\mathscr{N}$; $|\mathscr{M}|$ is the cardinal of $\mathscr{M}$. If $\mathscr{M} \subset \mathscr{N}$ and $k \in \mathscr{N}$, $\mathscr{M} + k$ denotes $\{m + k \mid m \in \mathscr{M}\}$. The "upper density" d^* is defined by

$$d^*(\mathscr{M}) = \limsup_k \{ |\mathscr{M} \cap \mathscr{N}_k| / (k+1) \},$$

and the "lower density" $d_*(\mathscr{M})$ is the corresponding lim inf; if they are equal, their common value is the "density" $d(\mathscr{M})$. Note that if $\mathscr{M}$ is finite then $d(\mathscr{M})$ exists and $= 0$. We list the following properties for reference later; they are well known and easily verified(4).

(1) $0 \leqq d_*(\mathscr{M}) \leqq d^*(\mathscr{M}) \leqq 1$.

(2) $d^*(\mathscr{M} + k) = d^*(\mathscr{M})$, and similarly for d_* and d.

(3) $d^*(\mathscr{M}) + d_*(\mathscr{N} - \mathscr{M}) = 1$.

(4) $d^*(\mathscr{M} \cup \mathscr{P}) \leqq d^*(\mathscr{M}) + d^*(\mathscr{P})$.

(5) $d_*(\mathscr{M} \cup \mathscr{P}) \geqq d_*(\mathscr{M}) + d_*(\mathscr{P})$ if $\mathscr{M} \cap \mathscr{P} = \varnothing$.

(6) $d_*(\mathscr{M} \cap \mathscr{P}) \geqq d_*(\mathscr{M}) + d_*(\mathscr{P}) - 1$.

(7) If $d(\mathscr{P}) = 0$ then $d^*(\mathscr{M} \cup \mathscr{P}) = d^*(\mathscr{M}) = d^*(\mathscr{M} - \mathscr{P})$, and similarly for d_* and d.

2.2. *Unit interval.* Letters like x, y denote points of the unit inteval I; thus $I = \{x \mid 0 \leqq x \leqq 1\}$. Capital letters A, B, C, etc., denote subsets of I. We take I with its usual topology and measure; the measure of A is written $\mu(A)$ or μA (all the sets we consider are measurable), and the closure of A is $\mathrm{Cl}(A)$.

(4) See [3, pp. 71–73].

The term "interval" is used to include every possible subinterval of I; it may be empty, or reduce to one point, and may contain 0, 1 or 2 endpoints. For the length of an interval J we use the efficient (but perhaps pompous) notation $\mu(J)$. An interval is "rational" if both its endpoints are rational.

2.3. *Sequences.* We shall have to operate freely with sequences of points of I, and so must be rather fussy with the notation for them. In this paper, a *sequence* is a function from a *subset* of $\mathcal{N}$ (perhaps $\mathcal{N}$ itself) to I. Sequences are denoted by letters like ξ, θ, ζ. The domain of a sequence ξ is written $\mathscr{D}(\xi)$, or $\mathscr{D}$ $(=\xi^{-1}(I))$; the range of ξ is $R(\xi) = \xi(\mathscr{D}(\xi)) \subset I$. We often write a sequence as $\{x_n \mid n \in \mathscr{D}\}$, meaning the function whose value at n is x_n. If ξ is defined on all of $\mathcal{N}$ (that is $\mathscr{D} = \mathcal{N}$), ξ is a "*full sequence*"; its range is then, of course, $\xi(\mathcal{N})$.

A *subsequence* of a sequence ξ is the restriction of ξ to a subset $\mathscr{M}$ of $\mathscr{D}(\xi)$. We take seriously the definition of a function as a set of ordered pairs; thus "θ is a subsequence of ξ" is equivalent to "$\theta \subset \xi$," and if θ, ξ are sequences then so is $\theta \cap \xi$. (In general $\theta \cup \xi$ is not a sequence, but it is one if both θ, ξ are subsequences of a third sequence.)

A sequence ξ is called "infinite" if its domain $\mathscr{D}$ is infinite; its range may be finite. If ξ is an infinite sequence, then lim sup ξ $(\in I)$ and lim inf ξ are defined as usual; if they are equal, their common value is lim ξ, and ξ "converges." To illustrate the notation, we mention explicitly the familiar fact:

(1) $x \in \mathrm{Cl}(R(\xi))$ if and only if either (a) $x \in R(\xi)$ or (b) there exists an infinite subsequence θ of ξ such that $\lim \theta = x$.

2.4. *Shifts.* Let ξ be a given 1-1 full sequence (*i.e.*, a 1-1 map of $\mathcal{N}$ in I; this is exactly what is usually called a "sequence of distinct points"). The map S_ξ (of $\xi(\mathcal{N})$ in I) whose value at $\xi(n)$ is $\xi(n+1)$ is called the "shift" with respect to ξ. It is well defined (and 1-1) because ξ is 1-1. A subset $\mathscr{M}$ of $\mathcal{N}$ is called a *shift extension set* (with respect to ξ) if there exists a *homeomorphism* S^*, of $\mathrm{Cl}(\xi(\mathscr{M}))$ onto some subset of I, such that for all $t \in \mathrm{Cl}(\xi(\mathscr{M}))$ and $n \in \mathcal{N}$ we have

$$(1) \qquad S^*(t) = \xi(n+1) \text{ if and only if } t = \xi(n).$$

It follows that S^* and S_ξ agree whenever both apply; in particular, $S^* \mid \xi(\mathscr{M}) = S_\xi \mid \xi(\mathscr{M})$. We refer to S^* as a "shift extension"; clearly S^* is uniquely determined by ξ and $\mathscr{M}$ (if it exists at all). Trivially, every finite $\mathscr{M} \subset \mathcal{N}$ is a shift extension set.

A routine verification (using the compactness of I) gives:

LEMMA 1. *A necessary and sufficient condition for $\mathscr{M}$ $(\subset \mathcal{N})$ to be a shift extension set with respect to a given* 1-1 *full sequence ξ, is that, whenever $m_0 < m_1 < m_2 < \cdots$ and $m_i \in \mathscr{M}$ $(i \in \mathcal{N})$, then both the following statements hold.*

(i) *The sequence $\{\xi(m_i) \mid i \in \mathcal{N}\}$ converges if and only if the sequence $\{\xi(m_i+1) \mid i \in \mathcal{N}\}$ converges.*

(ii) *For each* $n \in \mathcal{N}$, $\lim\{\xi(m_i) \mid i \in \mathcal{N}\} = \xi(n)$ *if and only if* $\lim\{\xi(m_i + 1) \mid i \in \mathcal{N}\} = \xi(n + 1)$.

3. **Uniformly distributed sequences.**

3.1. A full sequence ξ is said to be "uniformly distributed" in I provided that, for each interval J (see 2.2), the density $d(\xi^{-1}(J))$ exists and equals $\mu(J)$. If ξ is uniformly distributed, then its range $\xi(\mathcal{N})$ is clearly dense in I; further, it is well known (and easily proved) that

(1) $d(\xi^{-1}(A)) = \mu(A)$ whenever A is the union of a finite number of intervals.

It readily follows from (1) that, if C is a closed subset of I, then $d^*(\xi^{-1}(C)) \leqq \mu(C)$, and consequently

(2) if $\mathcal{M} \subset \mathcal{N}$, $d^*(\mathcal{M}) \leqq \mu(\mathrm{Cl}(\xi(\mathcal{M})))$.

3.2. The following lemma will be useful later (cf. 6.3).

Lemma 2. *Given a uniformly distributed sequence* ξ, *a countable subset* A *of* I, *and* $\varepsilon > 0$, *there exists* $\mathcal{M} \subset \mathcal{N}$ *such that* (i) $d_*(\mathcal{M}) \geqq 1 - \varepsilon$, (ii) *no point of* A *is a point of accumulation of* $\xi(\mathcal{M})$.

We may clearly suppose A to be infinite; enumerate it as $\{a_k \mid k \in \mathcal{N}\}$, and let J_k be an open interval around a_k of length $< \varepsilon / 2^{k+2}$. Thus $\sum_k \mu(J_k) < \varepsilon / 2$. Using 3.1(1) we define integers $m_0, m_1, \cdots$ such that $m_0 < m_1 < \cdots$ and, for all $n \geqq m_k$,

$$(1) \qquad \left|\xi^{-1}(J_0 \cup J_1 \cup \cdots \cup J_k) \cap \mathcal{N}_n\right| / (n + 1) < 2\mu(J_0 \cup \cdots \cup J_k) < \varepsilon.$$

We define $\mathcal{M}$ to be the set of all $n \in \mathcal{N}$ such that either (a) $n \leqq m_0$, or (b) for some $k \in \mathcal{N}$ we have both $m_k < n \leqq m_{k+1}$ and $\xi(n) \in I - (J_0 \cup \cdots \cup J_k)$. If $n \in \mathcal{M}$ is such that $\xi(n) \in J_k$, if follows that $n \leqq m_k$; hence the neighborhood J_k of a_k meets $\xi(\mathcal{M})$ in at most a finite set proving (ii). To prove (i), for each $n \in \mathcal{N}$ such that $n > m_0$ we define k_n to be the k such that $m_k < n \leqq m_{k+1}$. Then, if $r \in \mathcal{N} - \mathcal{M}$ and $r \leqq n$, we must have $\xi(r) \in J_0 \cup \cdots \cup J_k$ $(k = k_n)$. Thus $|(\mathcal{N} - \mathcal{M}) \cap \mathcal{N}_n| / (n + 1) < \varepsilon$, by (1), so that $d^*(\mathcal{N} - \mathcal{M}) \leqq \varepsilon$, and (i) follows.

3.2. Corollary. *Given a uniformly distributed sequence* ξ *and a countable subset* A *of* I, *there exist subsets* $\mathcal{M}_n$ $(n \in \mathcal{N})$ *of* $\mathcal{N}$ *such that* (i) $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots$, (ii) $d_*(\mathcal{M}_n) \to 1$ *as* $n \to \infty$, (iii) *for each* n, *no point of* A *is a point of accumulation of* $\xi(\mathcal{M}_n)$.

For, by applying Lemma 2 with $\varepsilon = 1/(n + 1)$, we obtain sets $\mathcal{M}'_n$ satisfying (ii) and (iii); we put $\mathcal{M}_n = \mathcal{M}'_0 \cup \cdots \cup \mathcal{M}'_n$.

4. **Transformations.**

4.1. *Notation.* We use the term "transformation" to mean a 1-1 mapping of a subset D of I in I, where $\mu(D) = 1$, and where T is measurable qua real function.

That is, for every Borel set $X \subset I$, $T^{-1}(X)$ is measurable. If further $T^{-1}(X)$ is measurable for every *measurable* $X \subset I$, T is said to be a "measurable transformation." The domain D of T is denoted by $D(T)$ when the dependence on T needs emphasis. Of course, $T(A)$ means $T(D \cap A)$; similarly $T^{-1}(A)$ means $T^{-1}(T(D) \cap A)$. As usual, two transformations T_1, T_2 are "equal a.e." if there exists $A \subset D(T_1) \cap D(T_2)$ such that $\mu(A) = 1$ and the restrictions $T_1 | A$, $T_2 | A$ coincide.

If T is a transformation and $x \in I$ is such that $T^n(x)$ is defined for all $n \in \mathcal{N}$, the sequence $\{T^n(x) | \ n \in \mathcal{N}\}$ is the "positive semi-orbit" of x under T; we shall, however, call it simply the *orbit* of x under T, for short, and we denote it by $\xi_{T,x}$, or just ξ_x when T can be understood. In other words, $\xi_x(n) = T^n(x)$ $(n \in \mathcal{N})$. Because T *is* 1-1, the orbit ξ_x (if it exists) is always a 1-1 full sequence. If $\mu(\bigcap\{T^{-n}(D(T)) \ | \ n \in \mathcal{N}\}) = 1$, then clearly ξ_x exists for almost all $x \in I$ (and conversely).

A transformation T is "measure-preserving" (m.-p. for short) if it is measureable and satisfies $\mu(A) = \mu(T^{-1}A)$ for all measurable $A \subset I$(5). An *ergodic* m.-p. transformation will be called "e.m.-p.". We note the familiar results(6):

(1) If T is m.-p., so is T^k $(k = 0, \ \pm 1, \ \pm 2, \cdots)$.

(2) If T is e.m.-p., so is T^{-1} (but T^n need not be).

(3) If $T_1 = T_2$ a.e., then if T_1 is m.-p. then if T_1 is m.-p., or e.m.-p., then so is T_2.

4.2. If T is a transformation such that $T \mid C$ is continuous, where C is compact, then (because T is 1-1) $T | C$ is a homeomorphism. Hence a straightforward application of Lusin's theorem gives:

Lemma 3. *Given a transformation T, there exist compact sets $C_n \subset D(T)$ $(n \in \mathcal{N})$ such that $C_0 \subset C_1 \subset \cdots$, $\mu(C_n) \to 1$ as $n \to \infty$, and $T | C_n$ is a homeomorphism for each n.*

4.3. The following result is essentially known, but we give the proof as it is simple and does not seem to be easily available(7).

Lemma 4. *If T is a transformation* (*in the sense of* 4.1), *the following statements are equivalent.*

(i) *T is m.-p.*

(ii) *For each rational closed interval J, $\mu(T^{-1}J) = \mu J$.*

(iii) *For each rational closed interval J, $T(J)$ is measurable and $\mu(TJ) = \mu(J)$.*

Proof. Trivially (i) $\Rightarrow$ (ii). We prove (ii) $\Rightarrow$ (i) as follows. Let $\mathfrak{B}$ denote the family of all Borel subsets of I; then, for each $X \in \mathfrak{B}$, $T^{-1}X$ is measurable. Thus the

(5) Some authors further require of an m.-p. transformation that TA is to be measurable whenever A is (and hence of the same measure as A). Under the present circumstances, the two definitions are equivalent [2, p. 584]; this also follows from Lemma 4 below.

(6) The only nontrivial step is proving (1) when $k = -1$; this is easy from Lemma 4 below.

(7) This is a sharpening of a special case of a theorem of von Neumann [2, p. 584].

function μ' defined by $\mu'(X) = \mu(T^{-1}X)$ is a countably additive measure defined on $\mathfrak{B}$. Assuming (ii), we have $\mu'(X) = \mu X$ whenever X is a closed rational interval. It follows that $\mu'(X) = \mu(X)$ for all $X \in \mathfrak{B}$. Moreover, if N is null, we have $N \subset N^*$ for some null G_δ set N^*; hence $\mu'(N) \leqq \mu'(N^*) = 0$. Since every measurable set A differs from a Borel set by a null set, we see that $T^{-1}(A)$ is measurable and $\mu'(A) = \mu(A)$, proving (i).

By interchanging T and T^{-1} in the preceding, we obtain a proof that (iii)$\Rightarrow$(i). Finally, we assume (i) and deduce (iii). We apply Lemma 3, and write $\bigcup_n C_n = E$. If $X \in \mathfrak{B}$, then each $X \cap C_n \in \mathfrak{B}$, and therefore $T(X \cap E)$ is a Borel set; hence $\mu(T(X \cap E)) = \mu(T^{-1}(T(X \cap E))$; that is, $\mu(TX \cap TE) = \mu(X \cap E)$. In particular, on taking $X = E$ we obtain $\mu(TE) = \mu(E) = 1$. Now if $X \in \mathfrak{B}$ we have that $TX \cap TE$ is Borel, and $TX - TE$ is null; hence TX is measurable and $\mu(TX) = \mu(TX \cap TE) = \mu(X \cap E) = \mu(X)$, establishing (iii). (Of course, the same argument as before then shows that (iii) also holds with J replaced by an arbitrary measurable set.)

4.4. We shall later need the following strengthened form of Lemma 3 for m.-p. transformations. (It would apply, more generally, to measurable transformations for which all the iterates are defined a.e.)

Lemma 5. *Given an m.-p. transformation T, there exist compact sets $C_n \subset D(T)$ $(n \in \mathcal{N})$ such that* (i) $C_n \cup TC_n \subset C_{n+1}$, (ii) $\mu(C_n) \to 1$ *as* $n \to \infty$, (iii) *for each* $n \in \mathcal{N}$, *and for all* $k = 0, \pm 1, \pm 2, \cdots$, $T^k | C_n$ *is a homeomorphism.*

Here T^k is defined a.e., and so is itself a transformation, for each k $(=0, \pm 1, \cdots)$. Applying Lemma 3 to T^k, we obtain compact sets K_{km}, $m \in \mathcal{N}$, such that $K_{k0} \subset K_{k1} \subset \cdots \subset D(T^k)$, $\mu(K_{km}) > 1 - 1/2^{2|k| + m}$, and $T^k | K_{km}$ is a homeomorphism. Define $L_m = \bigcap \{K_{km} | \ k = 0, \pm 1, \cdots\}$; then $T^k | L_m$ is also a homeomorphism, and $\mu(L_m) > 1 - 1/2^{m-2}$. Put $C_n = \bigcup \{T^i L_m \mid i, m \in \mathcal{N},\ i + m \leqq n\}$; this is clearly compact, and $C_n \subset D(T)$. Properties (i) and (ii) follow easily. To prove (iii) it suffices to prove that $T^k | C_n$ is continuous, for T^k is 1-1 and C_n is compact. Thus it suffices to prove that $T^k | T^i L_m$ is always continuous; but this is the composition of the inverse of the homeomorphism $T^i | L_m$ with the homeomorphism $T^{k+i} | L_m$.

4.5. We shall also need some criteria, mostly known, for the ergodicity of m.-p. transformations.

Lemma 6. *If T is an m.-p. transformation, the following statements are equivalent.*

(1) *T is ergodic.*

(2) *For each measurable $A \subset I$, and for almost all $x \in I$, $d(\xi_x^{-1}(A))$ exists and equals $\mu(A)$.*

(2′) *For almost all $x \in I$ the sequence ξ_x is uniformly distributed.*

(3) *If A, B are measurable subsets of I, then*

$$(n+1)^{-1} \sum_{i=0}^{n} \mu(A \cap T^i B) \to \mu(A)\mu(B) \quad \textit{as } n \to \infty .$$

(3′) *The same as* (3) *except that* A, B *are restricted to be closed rational intervals.*

(4) *For almost all* $x \in I$ *we have that, for all intervals* J, K *and for all* $i \in \mathcal{N}$, *the density* $d(\xi_x^{-1}(J \cap T^i K))$ *exists; and, denoting this density by* $d_i(x)$, *we have*

$$(n+1)^{-1} \sum_{i=0}^{n} d_i(x) \to \mu(J)\mu(K) \quad \textit{as } n \to \infty .$$

(4′) *The same as* (4) *except that* J, K *are restricted to be closed rational intervals.*

(5), (5′) *The same as* (4) *and* (4′) *except that we replace* $d(\xi_x^{-1}(J \cap T^i K))$ *throughout by* $d(\xi_x^{-1}(J) \cap \{\xi_x^{-1}(K) + i\})$.

Remark. The exceptional null set in (2) depends, of course, on A; but in the remaining statements the exceptional null sets are fixed in advance. In the last four statements, the densities called $d_i(x)$ are in fact independent of x, as the proof shows.

Proof. The equivalence of (1), (2), (3) is proved in [1, p. 77]. The implication (2) $\Rightarrow$ (2′) would be trivial except that the exceptional null set must be specified in advance. However, assuming (2), we can discard a fixed null set N such that, for all $x \in I - N$ and for every *rational* interval J, we have $d(\xi_x^{-1}(J)) = \mu(J)$. If now J is an arbitrary interval, consisting of more than one point (the contrary case is trivial), we take, for each $\varepsilon > 0$, rational intervals H, L such that $H \subset J \subset L$ and $\mu(L - H) < \varepsilon$; then

$$\mu J - \varepsilon < \mu H = d(\xi_x^{-1}(H)) \leqq d_*(\xi_x^{-1}(J)) \leqq d^*(\xi_x^{-1}(J)) < \mu J + \varepsilon$$

similarly, whence $d(\xi_x^{-1}(J))$ exists and equals μJ. Thus (2) implies (2′).

Trivially (3) $\Rightarrow$ (3′). The implication (3′) $\Rightarrow$ (3) is probably known; a proof is included in Lemma 7 below (8.1). The first five statements (1) $-$ (3′) are thus equivalent.

Next, (1) $\Rightarrow$ (4′). For, assuming (1), we then have (2); so, by discarding countably many null sets, we arrange that, for each pair of rational intervals J, K and each $i \in \mathcal{N}$, $d(\xi_x^{-1}(J \cap TK))$ exists and equals $\mu(J \cap T^i K)$. The desired conclusion now follows from (3).

Conversely, (4′) $\Rightarrow$ (2′); for (2′), restricted to *rational* intervals, is essentially the special case $K = I$ of (4′); and (2′) in full generality follows by the argument used above to prove (2) $\Rightarrow$ (2′).

It is easily checked that the two sets of integers $\xi_x^{-1}(J \cap T^i K)$, $\xi_x^{-1}(J) \cap \{\xi_x^{-1}K + i\}$, are "almost identical" in the sense: an integer $n \geqq i$ belongs to one of them if and only if it belongs to the other. (However, the first of these sets may contain integers $< i$.) For fixed i, it follows at once that if one set has a density then so does the other and the two densities are equal. Thus (4) and (5) are equivalent, and (4′) and (5′) are equivalent. The implication (4) ⇒ (4′) being trivial, we complete the proof of Lemma 6 by proving (2) ⇒ (4).

Assume (2), and therefore also (3); we can find a single null set N such that, for all $x \in I - N$, we have $d(\xi_x^{-1}(J \cap T^i K)) = \mu(J \cap T^i K)$ for all $i \in \mathscr{N}$ and for all *rational* intervals J, K. We deduce, by essentially the same argument as in the proof above that (2) ⇒ (2′), that this continues to hold for arbitrary intervals J, K; and (4) now follows immediately from (3).

5. *D*-sequences.

5.1. DEFINITION. A "D-sequence" ("determining sequence") is a 1-1 full sequence ξ with the following two properties:

(D1) To each $\varepsilon > 0$ corresponds $\delta > 0$ such that, whenever $\mathscr{M} \subset \mathscr{N}$ and $d_*(\mathscr{M}) > 1 - \delta$, then $\mu(\mathrm{Cl}(\xi(\mathscr{M}))) > 1 - \varepsilon$.

(D2) There exist shift extension sets (in the sense of 2.4) $\mathscr{M}_n$ $(n \in \mathscr{N})$ such that $\mathscr{M}_0 \subset \mathscr{M}_1 \subset \cdots$ and $d_*(\mathscr{M}_n) \to 1$ as $n \to \infty$.

We shall see later that "most" orbits under an ergodic measure-preserving transformation are D-sequences. Our first theorem asserts that a D-sequence determines (almost everywhere), in a natural way, a transformation T of which it is an orbit; we shall refer to T as the transformation "determined" by the D-sequence ξ, and denote it by T_ξ.

5.2. THEOREM 1. *Given a D-sequence ξ, there exists a transformation $T\,(=T_\xi)$ such that*

(i) $T(\xi(n)) = \xi(n+1)$ $(n \in \mathscr{N})$,

(ii) *For each $\varepsilon > 0$ there is a set $\mathscr{M} \subset \mathscr{N}$, of lower density $> 1 - \varepsilon$, such that, whenever $\xi(m_i) \to x$ and $m_i \in \mathscr{M}$ $(i \in \mathscr{N})$, then $\xi(m_i + 1) \to T(x)$.*

Further, T is essentially unique; if T' is another transformation satisfying (ii), *then $T = T'$ a.e.*

To construct T, let $\mathscr{M}_n$ $(n \in \mathscr{N})$ be as in (D2), and write $C_n = \mathrm{Cl}(\xi(\mathscr{M}_n))$. Since each $\mathscr{M}_n$ is a shift extension set, there exists a homeomorphism S_n^* of C_n (onto some subset of I) such that, for all $t \in C_n$ and $k \in \mathscr{N}$, we have $S_n^*(t) = \xi(k+1)$ if and only if $t = \xi(k)$. Now the separate homeomorphisms $S_0^*, S_1^*, \cdots,$ have nested domains, and (since $S_n^* | \mathscr{M}_k =$ shift $S_\xi | \mathscr{M}_k$ whenever $n \geqq k$) they agree where they overlap; hence they combine to a single-valued 1-1 map T of $\bigcup_n C_n$ in I. By hypothesis, $d_*(\mathscr{M}_n) \to 1$; hence, from (D1), we have $\mu(C_n) \to 1$, so that T is defined a.e. on I; and it is easily seen that T is a measurable function, and so a transformation. The construction gives $T(\xi(n)) = \xi(n+1)$ for all $n \in \bigcup_n \mathscr{M}_n$, and we extend T by defining $T(\xi(k)) = \xi(k+1)$ for all $k \in \mathscr{N} - \bigcup_n \mathscr{M}_n$; the verification that T remains 1-1 is immediate. Thus the assertion (i) of the theorem is obtained; and (ii) follows by taking $\mathscr{M} = \mathscr{M}_n$ with n large enough and noting that $T | C_n$ is continuous.

To prove uniqueness a.e., suppose that T is the particular transformation just constructed — we refer to this T as being "constructed from ξ and the sets $\mathscr{M}_n$" — and let T' be any transformation satisfying (ii). There exist, then, sets $\mathscr{M}'_n \subset \mathscr{N}$ $(n \in \mathscr{N})$ of lower density $\to 1$, and closed sets $C'_n = \mathrm{Cl}(\xi(\mathscr{M}'_n)) \subset I$, such that for each $n \in \mathscr{N}$ and $x \in I$ we have: if $\xi(m_i) \to x \in C'_n$ and $m_i \in \mathscr{M}'_n$ $(i \in \mathscr{N})$, then $\xi(m_i + 1) \to T'(x)$. Put $\mathscr{M}''_n = \mathscr{M}_n \cap \mathscr{M}'_n$, $C''_n = \mathrm{Cl}(\xi(\mathscr{M}''_n))$; from 2.1(6) we have $d_*(\mathscr{M}''_n) \to 1$, and from (D1) it follows that $\mu C''_n \to 1$. Now if $x \in C''_n$ there exists a sequence $\{\xi(m_i) \mid i \in \mathscr{N}\}$, with $m_i \in \mathscr{M}''_n$, converging to x; we then have $T'(x) = \lim \xi(m_i + 1) = T(x)$. Thus T' agrees with T on $\bigcup_n C''_n$ – that is, almost everywhere.

5.3. Corollary. *If two D-sequences differ only on a set of density* 0, *they determine the same transformation* (*a.e.*).

That is, if ξ and ξ' are D-sequences such that $\xi(n) = \xi'(n)$ for all $n \in \mathscr{N} - \mathscr{P}$, where $d(\mathscr{P}) = 0$, then $T_\xi = T_{\xi'}$ a.e. This is really a corollary to the method of proof of the uniqueness part of Theorem 1; we take $T' = T_{\xi'}$, replace $\mathscr{M}''_n$ by $\mathscr{M}''_n - \mathscr{P}$, and note that the argument still works.

5.4. Remark. If ξ is a D-sequence and ξ' is a 1-1 full sequence agreeing with ξ except on a set of density 0, one would expect ξ' to be necessarily a D-sequence; but I do not know whether this is always so. It is so if ξ is uniformly distributed, as follows from Theorem 3 below (6.3).

It would also be interesting to know just which transformations can be determined by suitable D-sequences. It is not hard to see that if a transformation T is determined by a D-sequence ξ then the range of T has measure 1; but the converse remains an open question.

6. Uniformly distributed D-sequences.

6.1. A uniformly distributed sequence ξ satisfies condition (D1) (5.1) automatically, from 3.1(2); hence it will be a D-sequence if and only if it is 1-1 and satisfies (D2). Such sequences exist, as will be shown later (6.4). On the other hand, not every D-sequence is uniformly distributed (6.5).

Since there is a close connection between ergodicity and uniform distribution (cf. Lemma 6, 4.5), one might expect that the transformation determined by a uniformly distributed D-sequence is necessarily e.m.-p. This is not quite correct, however; we show in the next theorem that it must be m.-p., but we give an example later (6.4) in which the transformation is emphatically not ergodic.

6.2. Theorem 2. *If ξ is a uniformly distributed D-sequence, the transformation T_ξ it determines is measure-preserving.*

We abbreviate T_ξ to T, and observe that, from Lemma 4 (4.3), it is enough to prove that, given a closed (rational) interval J, $T(J)$ is measurable and of measure μJ. We use condition (ii) of Theorem 1 to obtain sets $\mathscr{M}_n \subset \mathscr{N}$ $(n \in \mathscr{N})$ such that $d_*(\mathscr{M}_n) \to 1$ and, whenever $\xi(m_i) \to x$ $(m_i \in \mathscr{M}_n, n$ fixed), then $\xi(m_i + 1) \to Tx$).

From the fact that T is 1-1 and satisfies (i) of Theorem 1, we easily verify (using Lemma 1 (2.4)) that each $\mathcal{M}_n$ is a shift extension set. (In effect, we have just shown that every transformation T related to a D-sequence ξ as in Theorem 1, can be constructed from ξ as in the proof of Theorem 1; for it is easy to arrange that $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots$.) Thus $T \mid \mathrm{Cl}(\xi(\mathcal{M}_n))$ is a homeomorphism (coinciding with the shift extension). Write $\mathcal{P}_n = \xi^{-1}(J) \cap \mathcal{M}_n$, $K_n = \mathrm{Cl}(\xi(\mathcal{P}_n))$, $\lambda_n = d^*(\mathcal{N} - \mathcal{M}_n)$; thus, by 2.1(3), $\lambda_n = 1 - d_*(\mathcal{M}_n) \to 0$ as $n \to \infty$.

Since ξ is uniformly distributed, we have $\mu J = d(\xi^{-1}(J)) \leqq d^*(\mathcal{P}_n) + d^*(\xi^{-1}(J) - \mathcal{M}_n)$, by 2.1(4), and thus

$$\mu J \leqq d^*(\mathcal{P}_n) + \lambda_n. \tag{1}$$

Now $T \mid K_n$ is the shift extension homeomorphism; and an easy compactness argument then shows that $T(K_n) = \mathrm{Cl}(\xi(\mathcal{P}_n + 1))$. Hence (3.1(2)) $\mu(T(K_n)) \geqq d^*(\mathcal{P}_n + 1) = d^*(\mathcal{P}_n)$ from 2.1(2), and therefore the inner measure $\mu_*(T(J)) \geqq d^*(\mathcal{P}_n) \geqq \mu J - \lambda_n$, from (1). Making $n \to \infty$, we obtain

$$\mu_*(T(J)) \geqq \mu J, \quad \text{for every closed interval } J. \tag{2}$$

Now $\mathrm{Cl}(I - J)$ is the union of two disjoint closed intervals H_1, H_2 (possibly empty), to which the above also applies, giving

$$\mu_*(T(H_i)) \geqq \mu H_i \quad (i = 1, 2). \tag{3}$$

The intervals J, H_1, H_2 have only their endpoints in common, so $\mu^*(T(J)) = 1 - \mu_*(T(H_1) \cup T(H_2)) \leqq 1 - \mu_*(T(H_1)) - \mu_*(T(H_2)) \leqq 1 - \mu H_1 - \mu H_2 = \mu J$. This, with (2), gives that $T(J)$ is measurable and of measure μJ, as required.

6.3. Theorem 3. *If ξ is a uniformly distributed D-sequence, and ξ' is any* 1-1 *full sequence which agrees with ξ except on a set $\mathcal{P}$ of integers of density* 0, *then ξ' is also a uniformly distributed D-sequence, and the transformation $T_{\xi'}$ it determines agrees with T_ξ a.e.*

That ξ' is uniformly distributed is clear from the definition and 2.1(7). Hence we have only to verify that ξ' also satisfies (D2); for the assertion $T_{\xi'} = T_\xi$ a.e. will then follow from Theorem 1, Corollary (5.3).

Write $A = \xi(\mathcal{N}) \cup \xi'(\mathcal{N})$, and apply Lemma 2, Corollary (3.2); we obtain subsets $\mathcal{M}_n \subset \mathcal{N}$ $(n \in \mathcal{N})$ such that $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots, d_*(\mathcal{M}_n) \to 1$, and for each n no point of A is a point of accumulation of $\xi(\mathcal{M}_n)$. Again, since ξ satisfies (D2), there exist subsets $\mathcal{Q}_n$ of $\mathcal{N}$ $(n \in \mathcal{N})$ such that $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots$, $d_*(\mathcal{Q}_n) \to 1$, and each $\mathcal{Q}_n$ is a shift extension set for ξ. We write $\mathcal{E} = \{n \mid \xi(n) = \xi'(n)\} = \mathcal{N} - \mathcal{P}$; $\mathcal{F} = \{n \mid \xi(n+1) = \xi'(n+1)\}$; note that $d(\mathcal{E}) = d(\mathcal{F}) = 1$. Define $\mathcal{R}_n = \mathcal{M}_n \cap \mathcal{Q}_n \cap \mathcal{E} \cap \mathcal{F}$; clearly $\mathcal{R}_0 \subset \mathcal{R}_1 \subset \cdots$, and (from 2.1(6)) $d_*(\mathcal{R}_n) \to 1$ as $n \to \infty$. We shall verify that each $\mathcal{R}_n$ is a shift extension set for ξ'. From Lemma 1 (2.4) this comes to proving: if $m_0 < m_1 < \cdots$ and $m_i \in \mathcal{R}_n$ (n being fixed throughout), then

(1) $\xi'(m_i)$ converges if and only if $\xi'(m_i + 1)$ converges,

(2) if $k \in \mathscr{N}$, then $\xi'(m_i) \to \xi'(k)$ if and only if $\xi'(m_i + 1) \to \xi'(k + 1)$.

Now $\mathscr{R}_n \subset \mathscr{E} \cap \mathscr{F}$; hence $\xi'(m_i) = \xi(m_i)$ and $\xi'(m_i + 1) = \xi(m_i + 1)$ for all $m_i \in \mathscr{M}_n$, so (1) follows from Lemma 1 applied to the shift extension set $\mathscr{Q}_n$ for ξ, since $\mathscr{R}_n \subset \mathscr{Q}_n$. Finally, $\xi'(k)$ and $\xi'(k + 1)$ both $\in A$, and so (by construction of $\mathscr{M}_n$) condition (2) is satisfied vacuously.

Thus ξ' satisfies the condition (D2), and the proof is complete.

6.4. EXAMPLE 1. *There exists a uniformly distributed D-sequence which determines the identity transformation* (*a. e.*).

Roughly speaking, the example can be described by saying that, as n increases, $\xi(n)$ moves from 0 to 1 and back by steps of length $O(n^{-1/2})$. More precisely, for each $k = 1, 2, \cdots$ write $m_k = k(k + 1)$, so that $m_k - m_{k-1} = 2k$. If $m_{k-1} \leqq n < m_{k-1} + k$, take $\xi(n)$ somewhere between $(n - m_{k-1})/k$ and $(n - m_{k-1} + 1)/k$; if $m_{k-1} + k \leqq n < m_k$, take $\xi(n)$ between $(m_k - n - 1)/k$ and $(m_k - n)/k$; the values of $\xi(n)$ are chosen arbitrarily subject to these restrictions and to the further restriction that they are all different. Thus ξ is a 1-1 full sequence. We note that

(1) if $n \geqq m_{k-1}$ then $|\xi(n + 1) - \xi(n)| < 3/k$.

To see that ξ is uniformly distributed, let J be an interval; we easily check that, if n is large, the number of r's $\leqq n$ for which $\xi(r) \in J$ is $n\mu(J) + O(n^{1/2})$, so that $d(\xi^{-1}(J))$ exists and equals μJ, as required.

To verify that ξ is a D-sequence, it is enough to check (D2). By Lemma 2, Corollary (3.2) there exist sets $\mathscr{M}_0 \subset \mathscr{M}_1 \subset \cdots \subset \mathscr{N}$ such that $d_*(\mathscr{M}_n) \to 1$ and, for each n, no point of $\xi(\mathscr{N})$ is an accumulation point of $\xi(\mathscr{M}_n)$. It follows from Lemma 1 (2.4) that each $\mathscr{M}_n$ is a shift extension set for ξ; in fact, of the two conditions of this lemma, (i) holds from (1) above, while (as in the proof of Theorem 3 above) (ii) is vacuous from the choice of $\mathscr{M}_n$.

The transformation determined by ξ, as constructed from the sequence $\mathscr{M}_0, \mathscr{M}_1, \cdots$, is evidently the identity on each $\mathrm{Cl}(\xi(\mathscr{M}_n))$, because of (1). Thus $T_\xi =$ identity a.e.

REMARK. It would be interesting to know whether *every* m.-p. transformation can be determined (a.e.) by some uniformly distributed D-sequence.

6.5. The next example shows that the converse of Theorem 2 is false.

EXAMPLE 2. *There exists a D-sequence which determines the identity transformation* (*a.e.*), *and which is not uniformly distributed.*

Let ξ be the uniformly distributed D-sequence of Example 1, determining the identity transformation (a.e.). Let ϕ be a homeomorphism of I onto itself for which the ratios $\mu(\phi(A))/\mu(A)$ are bounded above and away from zero (but not all 1) — for instance $\phi(x) = x/2$ for $0 \leqq x \leqq 2/3$, $\phi(x) = 2x - 1$ for $2/3 \leqq x \leqq 1$. The sequence $\zeta = \phi\,\xi$ (that is, $\zeta(n) = \phi\,(\xi(n))$) is then a D-sequence determining the identity transformation (a.e.), and is not uniformly distributed.

7. Ergodic transformations and sequences.

7.1. DEFINITION. A 1-1 full sequence ξ is an "E-sequence" ("ergodic measure-preserving") providing it satisfies the following two conditions. (Recall that $\mathscr{M} + i$ denotes $\{n + i \mid n \in \mathscr{M}\}$.)

(E1) If J, K are closed rational intervals (in I), and $i \in \mathscr{N}$, the set $\xi^{-1}(J) \cap (\xi^{-1}(K) + i)$ has a density $d_i(J,K)$, and

$$(n+1)^{-1} \sum_{i=0}^{n} d_i(J,K) \to \mu(J)\mu(K) \text{ as } n \to \infty .$$

(E2) There exist subsets $\mathscr{M}_n \subset \mathscr{N}$ $(n \in \mathscr{M})$ such that

(i) $\bigcup_n \mathscr{M}_n = \mathscr{N}$, and $\mathscr{M}_0 \subset \mathscr{M}_1 \subset \cdots$,

(ii) $\mathscr{M}_n + 1 \subset \mathscr{M}_{n+1}$ $(n \in \mathscr{N})$,

(iii) $d_*(\mathscr{M}_n) \to 1$ as $n \to \infty$,

(iv) each $\mathscr{M}_n$ is a shift extension set for ξ.

7.2. THEOREM 4. *Every E-sequence is a uniformly distributed D-sequence, and determines an e.m.-p. transformation.*

Let ξ be an E-sequence. Then it satisfies (E1) with $K=I$, and hence is uniformly distributed (from which (D1) follows). Also (E2) clearly implies (D2); thus ξ is a D-sequence. Consider the transformation $T = T_\xi$ it determines; it is m.-p., by Theorem 2 (6.2), and we must prove it ergodic.

Let J, K be closed rational intervals, and let $i \in \mathscr{N}$.
We shall prove

(1) $d_i(J,K) = \mu(J \cap T^i K)$.

Once (1) is established, (E1) gives $(n+1)^{-1} \sum_{i=0}^{n} \mu(J \cap T^i K) \to \mu(J)\mu)(K)$ as $n \to \infty$, and the ergodicity of T follows from Lemma 6 (4.5).

We may assume T is constructed (as in Theorem 1) from ξ and the sets $\mathscr{M}_n$ arising in (E2). Put $C_n = \mathrm{Cl}(\xi(\mathscr{M}_n))$; then $T \mid C_n$ is a homeomorphism for each n $(\in \mathscr{N})$. Also, from (E2) (ii), $T(C_n) \subset C_{n+1}$ for $T\xi(m)) = \xi(m+1)$.
It follows by an easy induction argument that $T^i(C_n) \subset C_{n+i}$ $(i \in \mathscr{N})$, and that

(2) $T^i \mid C_n$ is a homeomorphism $(i, n \in \mathscr{N})$.

Fixing i, we write

$$\mathscr{P} = \xi^{-1}(J), \quad \mathscr{Q} = \xi^{-1}(K), \mathscr{R} = \mathscr{P} \cap (\mathscr{Q} + i),$$

$$\mathscr{P}_n = \mathscr{P} \cap \mathscr{M}_n, \mathscr{Q}_n = \mathscr{Q} \cap \mathscr{M}_n, \ \mathscr{R}_n = \mathscr{P}_n \cap (\mathscr{Q}_n + i);$$

then $\mathscr{R} - \mathscr{R}_n \subset (\mathscr{N} - \mathscr{M}_n) \cup ((\mathscr{N} - \mathscr{M}_n) + i)$, of upper density (say) λ_n. Using 2.1, we see that (because $d_*(\mathscr{M}_n) \to 1) \lambda_n \to 0$ as $n \to \infty$. The density $d_i(J,K)$ occurring in (1) (and (E1)) is just $d(\mathscr{R})$, so we have

$$d_i(J,K) - \lambda_n \leqq d^*(\mathscr{R}_n) \leqq \mu(\mathrm{Cl}(\xi(\mathscr{R}_n))) \text{ by } 3.1(2)$$

$$\leqq \mu(\mathrm{Cl}(\xi(\mathscr{P}_n)) \cap \mathrm{Cl}(\xi(\mathscr{Q}_n + i))) \leqq \mu(J \cap A),$$

where $A = \mathrm{Cl}(\xi(\mathscr{Q}_n + i)) = \mathrm{Cl}(T^i(\xi(\mathscr{Q}_n))) \subset T^i\mathrm{Cl}(\xi(\mathscr{Q}_n))$ (from (2)) $\subset T^iK$. Thus we have

$$d_i(J,K) - \lambda_n \leqq \mu(J \cap T^iK),$$

and, making $n \to \infty$, we obtain

(3) $d_i(J,K) \leqq \mu(J \cap T^iK)$.

Now $I - J$ is the union of two intervals (possibly empty); call their closures H_1, H_2. The preceding argument applies with H_1, H_2 replacing J, giving

$$d_i(H_j,K) \leqq \mu(H_j \cap T^iK) \quad (j = 1,2).$$

Adding these inequalities to (3), and using 2.1(4), we obtain $d(\xi^{-1}(K)+i) \leqq \mu(T^iK)$; that is (since T is m.-p.)

(4) $d(\xi^{-1}(K)) \leqq \mu(K)$.

Now the same argument applies to the closures L_1, L_2 of the complementary intervals of K. Thus $d(\xi^{-1}(L_j)) \leqq \mu(L_j)$ $(j = 1,2)$. Adding these inequalities to (4) yields $1 \leqq 1$; since there is equality here, all the preceding inequalities must also be equalities, and (3) gives (1), completing the proof.

7.3. THEOREM 5. *If T is an e.m.-p. transformation, then for almost all $x \in I$ the orbit ξ_x $(= \{T^n(x) \mid n \in \mathscr{N}\})$ is an E-sequence, and the transformation it determines coincides a.e. with T.*

By Lemma 5 (4.4) there exist compact sets C_n $(n \in \mathscr{N})$ such that $C_{n+1} \supset TC_n \cup C_n$, $\mu C_n \to 1$ as $n \to \infty$, and $T^k \mid C_n$ is a homeomorphism $(k = 0, \pm 1, \cdots, n \in \mathscr{N})$. Put $F = \bigcup\{C_n \mid n \in \mathscr{N}\}$, $B = \bigcap\{T^kF \mid k = 0, \pm 1, \cdots\}$; then $I - B$ is null, and B is invariant under T. Applying Lemma 6(2) (4.5) to C_n, we obtain a null set N_n such that, for all $x \in I - N_n$, $d(\xi_x^{-1}(C_n))$ exists and equals μC_n. And, applying Lemma 6(5), we obtain a null set N such that, for all $x \in I - N$ and for all intervals J, K and all $i \in \mathscr{N}$, the density $d_i(x) = d(\xi_x^{-1}(J) \cap (\xi_x^{-1}(K) + i))$ exists, and further $(n+1)^{-1} \sum_{i=0}^n d_i(x) \to \mu(J)\mu(K)$ as $n \to \infty$.

Let x be an arbitrary point (fixed throughout what follows) of $B - N - \bigcup\{N_n \mid n \in \mathscr{N}\}$. We abbreviate ξ_x to ξ, and prove first that ξ is an E-sequence. It is clearly 1-1 (because T is) and defined on all $\mathscr{N}$ (because $x \in B$). Condition (E1) is immediate because $x \notin N$. To prove (E2), we put $\mathscr{M}_n = \xi^{-1}(C_n)$ $(n \in \mathscr{N})$; clearly $\mathscr{M}_0 \subset \mathscr{M}_1 \subset \cdots$, and $\bigcup_n \mathscr{M}_n = \mathscr{N}$ because $\xi(\mathscr{N}) \subset F$ (since $x \in B$). Also $d(\mathscr{M}_n)$ exists and equals μC_n, hence $\to 1$ as $n \to \infty$. The fact that $TC_n \subset C_{n+1}$ gives $\mathscr{M}_n + 1 \subset \mathscr{M}_{n+1}$. Finally, to show that $\mathscr{M}_n$ is a shift extension set for ξ, we verify that T gives the required shift extension on $\mathrm{Cl}(\xi(\mathscr{M}_n))$. In fact, $T(\xi(m)) = \xi(m+1)$ for all $m \in \mathscr{N}$, and $T \mid \mathrm{Cl}(\xi(\mathscr{M}_n))$ is a restriction of $T \mid C_n$, and so is a homeomorphism. This proves not only that ξ is an E-sequence, but also that the transformation T_ξ it determines (as constructed from the sets $\mathscr{M}_n$) coincides with T on $\bigcup\{\mathrm{Cl}(\xi(\mathscr{M}_n)) \mid n \in \mathscr{N}\}$. Now, the sequence ξ being uniformly distributed (from Theorem 4, 7.2), we have from 3.1(2) that $\mu(\mathrm{Cl}(\xi(\mathscr{M}_n))) \geqq d(\mathscr{M}_n) = \mu C_n \to 1$; hence $T_\xi = T$ a.e.

REMARK. We have proved a little more than that ξ_x is an E-sequence; we have shown that it satisfies the strengthened form of (E2) in which we require that the density $d(\mathscr{M}_n)$ exists. I do not know whether every E-sequence satisfies this strengthened condition.

7.4. Theorem 5 answers the main question considered in this paper, by showing that every e.m.-p. transformation is determined, in a natural way, by its "general" orbit. However, the method of determination here, based on the construction of Theorem 1, is somewhat complicated, and it may be asked whether these complications are necessary. In particular the following simplified rule may seem plausible. Let us say that a transformation T is "generated" by a (full) sequence ξ providing there exists $\mathscr{M} \subset \mathscr{N}$, of upper density 1, and a null set $N \subset I$, such that: whenever $m_0 < m_1 < \cdots$ and $m_i \in \mathscr{M}\,(i \in \mathscr{N})$, and $\xi(m_i) \to y \in I - N$, then $\xi(m_i + 1) \to T(y)$. One might hope that every e.m.-p. transformation T would be "generated," in this sense, by its general orbit. But this hope is refuted by the following counterexample.

EXAMPLE. 3. *There exists an e.m.-p. transformation T which, for almost all $x \in I$, fails to be "generated" by the orbit ξ_x of x.*

To see this, we first show that if T is "generated" by a sequence ξ, as above, and if X denotes the set of all accumulation points of the set $\xi(\mathscr{M})$, then $T|X - N$ *must be continuous.* To prove this, it is enough to show that if $y_n \to y$ as $n \to \infty$, where y and each y_n are points of $X - N$, then $T(y_n) \to T(y)$. Because $y_n \in X$, there exist integers $m_{n1} < m_{n2} < \cdots$ such that $m_{ni} \in \mathscr{M}$ and $\xi(m_{ni}) \to y_n$ as $i \to \infty$; and it follows that $\xi(m_{ni} + 1) \to T(y_n)$. By picking out a suitable diagonal sequence, we obtain integers $m_1' < m_2' < \cdots$ in $\mathscr{M}$ (where $m_n' = m_{ni}$ for a suitable large enough i) such that $|\xi(m_n') - y_n| \to 0$ and $|\xi(m_n' + 1) - T(y_n)| \to 0$ as $n \to \infty$. It follows that $\xi(m_n') \to y$, so that $\xi(m_n' + 1) \to T(y)$, and therefore $T(y_n) \to T(y)$, as required.

In particular, if T is an e.m.-p. transformation, and if T is "generated" by the orbit ξ_x for a non-null set of x's, we can find such an x for which ξ_x is uniformly distributed (Lemma 6, 4.5). Since $\mathscr{M}$ has upper density 1, the set of accumulation points of $\xi_x(\mathscr{M})$ is all of I. Thus *T must be continuous on $I - N$, where N is null.*

On the other hand, by standard techniques one can construct an e.m.-p. transformation T which is *not* continuous when restricted to the complement of any null set; and this provides the desired example. (One method of construction for such a T can be sketched as follows. We begin by constructing an m.-p. transformation ϕ of the interval $J = [0, 1/2)$ onto the interval $K = [1/2, 1]$, such that, for every null set $N \subset J$, $\phi|J - N$ is discontinuous. For example, take complementary sets C, D in J which have positive measure in every nontrivial interval of J, and define

$$\phi(x) = 1/2 + \mu(C \cap [0, x]) \text{ if } x \in C,$$
$$= 1 - \mu(D \cap [x, 1/2]) \text{ if } x \in D.$$

Now let ψ be any e.m.-p. transformation of J onto itself, and define

$$T(x) = \phi(x) \text{ if } x \in J,$$
$$= \psi(\phi^{-1}(x)) \text{ if } x \in K.$$

Clearly T is an m.-p. transformation of I onto itself, and it can be seen to be ergodic.

8. **Mixing transformations.**

8.1. Let T be a transformation (on I), and let A, B be arbitrary measurable subsets of I. It follows from Lemmas 6 (4.5) and 4 (4.3) that T is e.m.-p. if and only if (for all A, B)

(a) $$(n+1)^{-1} \sum_{i=0}^{n} \mu(A \cap T^iB) \to \mu(A)\mu(B) \quad \text{as } n \to \infty.$$

We recall that T is said to be "weakly mixing" if (for all A, B)

(b) $$(n+1)^{-1} \sum_{i=0}^{n} \left|\mu(A \cap T^iB) - \mu(A)\mu(B)\right| \to 0 \quad \text{as } n \to \infty,$$

and "strongly mixing" if (for all A, B)

(c) $$\mu(A \cap T^nB) \to \mu(A)\mu(B) \quad \text{as } n \to \infty.$$

Clearly, strongly mixing $\Rightarrow$ weakly mixing $\Rightarrow$ ergodic measure-preserving. We have already seen, in Lemma 6, that the ergodicity of T can be characterized in terms of properties of the "general" orbit under T; our object now is to obtain similar results for the weak and strong mixing properties. First we need a lemma, which incidentally completes the proof of Lemma 6.

LEMMA 7. *Let T be an m.-p. transformation. Then, for each of the above statements* (a), (b), (c), *if it holds whenever A, B are closed rational intervals, it also holds whenever A, B are measurable subsets of I.*

We give the proof for (a); the argument for (b) and (c) is entirely similar. We use the notation $X + Y$ for the symmetric difference of the sets X, Y (that is, $(X - Y) \cup (Y - X)$).

Given arbitrary measurable A, B, and $\varepsilon > 0$, there exist sets X, Y, both of which are unions of finitely many closed rational intervals, such that $\mu(A + X) < \varepsilon$ and $\mu(B + Y) < \varepsilon$. For each $i \in \mathcal{N}$, $(A \cap T^iB) + (X \cap T^iY) \subset (A + X) \cap T^i(B + Y)$, and therefore $\left|\mu(A \cap T^iB) - \mu(X \cap T^iY)\right| \leqq \mu(A + X) + \mu(T^i(B + Y)) < 2\varepsilon$. Hence $(n+1)^{-1}\sum_{i=0}^{n}\mu(A \cap T^iB)$ differs from $(n+1)^{-1}\sum_{i=0}^{n}\mu(X \cap T^iY)$ by $< 2\varepsilon$. Also $\mu(A)\mu(B)$ differs from $\mu(X)\mu(Y)$ by $O(\varepsilon)$. Now assume that (a) holds for closed rational intervals; then it also holds for X and Y, so that $(n+1)^{-1}\sum_{i=0}^{n}\mu(A \cap T^iB)$ differs from $\mu(A)\mu(B)$ by $O(\varepsilon)$ if n is large enough, giving the result.

8.2. Before stating the next theorem, we recall the notation $d_i(J, K)$ occurring in the definition of an E-sequence; see (E1), 7.1.

THEOREM 6. *Let ξ be an E-sequence, and T_ξ the transformation it determines. Then*

(1) *T_ξ is weakly mixing if and only if ξ satisfies the condition:*

(WM) *For all closed rational intervals J, K,*

$$(n+1)^{-1} \sum_{i=0}^{n} \left| d_i(J, K) - \mu(J)\mu(K) \right| \to 0 \quad \text{as } n \to \infty\,;$$

(2) *T_ξ is strongly mixing if and only if ξ satisfies the condition:*

(SM) *For all closed rational intervals J, K,*

$$d_i(J, K) \to \mu(J)\mu(K) \quad \text{as } n \to \infty\,.$$

Proof. In proving that T_ξ is ergodic, we have shown (7.2 (1)) that $d_i(J, K) = \mu(J \cap T^i K)$. Thus the theorem is an immediate consequence of the definitions and Lemma 7.

8.3. COROLLARY. *If T is a measure-preserving transformation (of I) then*

(1) *T is ergodic if and only if, for almost all $x \in I$, the orbit ξ_x of x under T is an E-sequence;*

(2) *T is weakly mixing if and only if, for almost all $x \in I$, ξ_x is an E-sequence satisfying* (WM);

(3) *T is strongly mixing if and only if, for almost all $x \in I$, ξ_x is an E-sequence satisfying* (SM).

For if T is e.m.-p., we have seen (Theorem 5, 7.3) that ξ_x is an E-sequence (for almost all x). Conversely, if ξ_x is (a.e.) an E-sequence, it is uniformly distributed (Theorem 4, 7.2), and hence (Lemma 6, 4.5) T is ergodic. The assertions (2) and (3) now follow from Theorem 6, in view of the fact that (a.e.) if $\xi = \xi_x$ then $T_\xi = T$ (Theorem 5).

REMARK. We could replace the phrase "an E-sequence" by "uniformly distributed" throughout this corollary. For the proof of (1) shows that, if almost all the orbits under an m.-p. transformation are uniformly distributed, they are E-sequences.

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UNIVERSITY OF ROCHESTER,
ROCHESTER, NEW YORK